Contexto: Mecânica Newtoniana

- Dinâmica de partícula:

$$\vec{F}_R = \frac{d\vec{p}}{dt} = \dot{\vec{p}} \;, \qquad \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} = m\dot{\vec{r}}$$

- Dinâmica de sistemas de partículas:

$$\vec{F}_{R_i} = \vec{F}_{ext_i} + \sum_{j\neq i} \vec{F}_{ji} = \dot{\vec{p}}_i = \frac{d}{dt}(m_i\dot{\vec{r}}_i)$$

HIPÓTESE 1: $\dot{m}_i = 0$ $\Rightarrow$ $\vec{F}_{ext_i} + \sum_{j\neq i} \vec{F}_{ji} = m_i\ddot{\vec{r}}_i$ $(*_i)$

$$\sum_i (*)_i \Rightarrow \sum_i \vec{F}_{ext_i} + \underbrace{\sum_i\sum_{j\neq i} \vec{F}_{ji}}_{(\vec{F}_{21}+\vec{F}_{31}+\dots)+(\vec{F}_{12}+\vec{F}_{32}+\dots)+(\vec{F}_{13}+\vec{F}_{23}+\dots)=\vec{0}} = \sum_i m_i\ddot{\vec{r}}_i$$

$$\Rightarrow \sum_i \vec{F}_{ext_i} = \sum_i m_i\ddot{\vec{r}}_i \overset{(H1)}{=} \frac{d^2}{dt^2}\left(\sum_i m_i\vec{r}_i\right) = M\frac{d^2}{dt^2}\left(\frac{\sum_i m_i\vec{r}_i}{M}\right)$$

$$\Rightarrow \boxed{\vec{F}_{R_{ext}} = M\ddot{\vec{R}} = \dot{\vec{P}}}$$

, onde $M = \sum_i m_i$,

$$\vec{R} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \quad \text{e}$$

$$\vec{P} = \sum_i \vec{p}_i$$

$$\vec{L}_a = \sum_i \vec{r}_{ia} \times \vec{p}_{ia}:$$

$$\dot{\vec{L}}_a = \frac{d}{dt}\left(\sum_i \vec{r}_{ia} \times \vec{p}_{ia}\right)$$

$$= \sum_i \left(\frac{d\vec{r}_{ia}}{dt} \times \vec{p}_{ia} + \vec{r}_{ia} \times \frac{d\vec{p}_{ia}}{dt}\right)$$

$$= \sum_i \left(\dot{\vec{r}}_{ia} \times \vec{p}_{ia} + \vec{r}_{ia} \times \dot{\vec{p}}_{ia}\right)$$

$$\overset{(H1)}{=} \sum_i \left(\vec{0} + (\vec{r}_i - \vec{r}_a) \times (\dot{\vec{p}}_i - m_i \ddot{\vec{r}}_a)\right)$$

$$= \sum_i \left((\vec{r}_i - \vec{r}_a) \times \left(\sum_{j\neq i} \vec{F}_{ji} + \vec{F}_{ext_i}\right) - m_i (\vec{r}_i - \vec{r}_a) \times \ddot{\vec{r}}_a\right)$$

$$= \sum_i \left(\sum_{j\neq i} (\vec{r}_i - \vec{r}_a) \times \vec{F}_{ji} + (\vec{r}_i - \vec{r}_a) \times \vec{F}_{ext_i} - m_i (\vec{r}_i - \vec{r}_a) \times \ddot{\vec{r}}_a\right)$$

$$= \underbrace{\sum_i \sum_{j\neq i} (\vec{r}_i - \vec{r}_a) \times \vec{F}_{ji}}_{} + \underbrace{\sum_i (\vec{r}_i - \vec{r}_a) \times \vec{F}_{ext_i}}_{= \vec{G}_{ext,a}} - \underbrace{\sum_i m_i \vec{r}_i \times \ddot{\vec{r}}_a}_{M\vec{R} \times \ddot{\vec{r}}_a} + \underbrace{\sum_i m_i \vec{r}_a \times \ddot{\vec{r}}_a}_{M\vec{r}_a \times \ddot{\vec{r}}_a}$$

$$= (\vec{r}_1 - \vec{r}_a) \times \vec{F}_{21} + (\vec{r}_1 - \vec{r}_a) \times \vec{F}_{31} + \dots$$

$$+ (\vec{r}_2 - \vec{r}_a) \times \vec{F}_{12} + (\vec{r}_2 - \vec{r}_a) \times \vec{F}_{32} + \dots$$

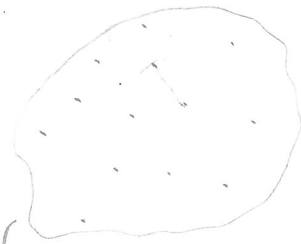
$$+ (\vec{r}_3 - \vec{r}_a) \times \vec{F}_{13} + (\vec{r}_3 - \vec{r}_a) \times \vec{F}_{23} + \dots$$

$$+ \dots$$

$$= (\vec{r}_1 - \vec{r}_a - \vec{r}_2 + \vec{r}_a) \times \vec{F}_{21} + (\vec{r}_1 - \vec{r}_a - \vec{r}_3 + \vec{r}_a) \times \vec{F}_{31} + (\vec{r}_2 - \vec{r}_a - \vec{r}_3 + \vec{r}_a) \times \vec{F}_{32} + \dots$$

$$\Rightarrow \quad \dot{\vec{L}}_a = \sum_i \sum_{j>i} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} + \vec{\tau}_{ext,a} - M(\vec{R} - \vec{r}_a) \times \ddot{\vec{r}}_a$$

HIPÓTESE 2: $(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = \vec{0} \quad \forall i,j$



↳ "Forma forte da 3ª lei de Newton" ⟶ p/ corpo rígido: componentes de cisalhamento não satisfazem!

Sob (H2), logo:

$$\dot{\vec{L}}_a = \vec{\tau}_{ext,a} - \underbrace{M(\vec{R} - \vec{r}_a) \times \ddot{\vec{r}}_a}_{= \vec{0} \text{ se } \vec{r}_a = \vec{R} \text{ ou } \dot{\vec{r}}_a \text{ é cte.}}$$

centro de massa parece relevante,

ou no mínimo especial... ⟶

Ops :

$$\vec{L}_a = \sum_i (\vec{r}_i - \vec{r}_a) \times \left(m_i (\dot{\vec{r}}_i - \dot{\vec{r}}_a) \right)$$

$$= \sum_i (\vec{r}_i - \vec{R} + \vec{R} - \vec{r}_a) \times \left(m_i (\dot{\vec{r}}_i - \dot{\vec{R}} + \dot{\vec{R}} - \dot{\vec{r}}_a) \right)$$

$$= \underbrace{\sum_i (\vec{r}_i - \vec{R}) \times m_i (\dot{\vec{r}}_i - \dot{\vec{R}})}_{\sum_i \vec{r}_{i,cm} \times m_i \dot{\vec{r}}_{i,cm}} + \underbrace{\sum_i (\vec{r}_i - \vec{R}) \times m_i (\dot{\vec{R}} - \dot{\vec{r}}_a)}_{\sum_i m_i (\vec{r}_i - \vec{R}) \times (\dot{\vec{R}} - \dot{\vec{r}}_a)} + \overbrace{\sum_i (\vec{R} - \vec{r}_a) \times m_i (\dot{\vec{r}}_i - \dot{\vec{R}}}^{(\vec{R} - \vec{r}_a) \times (\sum_i m_i \dot{\vec{r}}_i - \sum_i m_i \ldots}$$

$$= \left(\sum_i m_i \vec{r}_i - M\vec{R} \right) \times (\dot{\vec{R}} - \dot{\vec{r}}_a)$$

$$= (M\vec{R} - M\vec{R}) \times (\dot{\vec{R}} - \dot{\vec{r}}_a)$$

$$= \vec{0}$$

$$\overset{(M1)}{=} (\vec{R} - \vec{r}_a) \times (M\dot{\vec{R}} - M \ldots$$

$$= \vec{0}$$

$$+ \underbrace{\sum_i (\vec{R} - \vec{r}_a) \times m_i (\dot{\vec{R}} - \dot{\vec{r}}_a}_{\sum_i m_i (\vec{R} - \vec{r}_a) \times (\dot{\vec{R}} - \dot{\vec{r}}_a}$$

$$\therefore \quad \vec{L}_a = \underbrace{\sum_i \vec{r}_{i,cm} \times \vec{p}_{i,cm}}_{\sum_i \vec{L}_{i,cm}} + \underbrace{(\vec{R} - \vec{r}_a) \times (\vec{P} - M\dot{\vec{r}}_a)}_{\vec{L}_{cm,a}}$$

- Trabalho de uma força:

$$W_{F_{a,b}} = \int_a^b \vec{F} \cdot d\vec{r}$$

- Trabalho resultante sobre um sistema de partículas:

$$W_{a,b} = \sum_i \int_a^b \vec{F}_{R_i} \cdot d\vec{r}_i$$

$$= \sum_i \int_a^b \frac{d\vec{p}_i}{dt} \cdot d\vec{r}_i = \sum_i \int_a^b d\vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$$

$$= \sum_i \int_a^b m_i \underbrace{d\vec{v}_i \cdot \vec{v}_i}$$

$$d(\vec{A} \cdot \vec{B}) = d\vec{A} \cdot \vec{B} + \vec{A} \cdot d\vec{B}$$

$$= \sum_i \int_a^b m_i \frac{1}{2} d(\vec{v}_i \cdot \vec{v}_i)$$

$$\overset{(H1)}{=} \sum_i \int_a^b d\left(\frac{m_i v_i^2}{2}\right) = \sum_i \underbrace{\frac{m_i v_i^2}{2}}_{T_i}\Bigg|_a^b = \left[\sum_i T_i\right]_a^b$$

$$\therefore \quad \underline{W_{a,b} = T_b - T_a}, \quad \text{onde} \quad T = \sum_i \frac{m_i \dot{\vec{r}}_i^{\,2}}{2} = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i}$$

"Teorema trabalho-energia cinética"

• Forças conservativas:

$$\boxed{\exists\, V(\vec{r}) : \vec{F} = -\vec{\nabla} V} \Longleftarrow \left\{ \begin{array}{c} \oint_C \vec{F} \cdot d\vec{r} = 0 \quad \forall\, C \text{ fechada} \\ \Updownarrow \\ \vec{\nabla} \times \vec{F} = \vec{0} \end{array} \right.$$

Assim:

$$W_{F_{a,b}} = \int_a^b \vec{F} \cdot d\vec{r} = -\int_a^b \vec{\nabla} V \cdot d\vec{r} = -\int_a^b dV = -V\Big|_a^b = V_a - V_b$$

* Trabalho conservativo sobre um sistema de partículas:

$$W_{a,b} = W_{cons_{a,b}} + W_{\tilde{n}cons_{a,b}} = T_b - T_a$$

$$= \sum_i \int_a^b \vec{F}_{cons_i} \cdot d\vec{r} + W_{\tilde{n}cons_{a,b}} = T_b - T_a$$

$$= \sum_i (V_{i_a} - V_{i_b}) + W_{\tilde{n}cons_{a,b}} = T_b - T_a$$

Sendo $V = \sum_i V_i$, então:

$$\boxed{(T_b + V_b) - (T_a + V_a) = W_{\tilde{n}cons_{a,b}}}$$

Esse V inclui tanto forças externas como internas!

HIPÓTESE 3 : $V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|)$.

Se $V_{ij} = V_{ij}(s_{ij}^2)$, onde $s_{ij}^2 = |\vec{r}_i - \vec{r}_j|^2$

$$= (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2,$$

então

$$\vec{F}_{ji} = -\vec{\nabla}_i V_{ij}$$

$$= -\hat{\imath}\, \frac{\partial V_{ij}}{\partial s_{ij}^2} \frac{\partial s_{ij}^2}{\partial x_i} - \hat{\jmath}\, \frac{\partial V_{ij}}{\partial s_{ij}^2} \frac{\partial s_{ij}^2}{\partial y_i} - \hat{k}\, \frac{\partial V_{ij}}{\partial s_{ij}^2} \frac{\partial s_{ij}^2}{\partial z_i}$$

$$= -\frac{\partial V_{ij}}{\partial s_{ij}^2} \left(\hat{\imath} \cdot 2(x_i - x_j) + \hat{\jmath} \cdot 2(y_i - y_j) + \hat{k} \cdot 2(z_i - z_j) \right)$$

$$= \left(-2 \frac{\partial V_{ij}}{\partial s_{ij}^2}\right) (\vec{r}_i - \vec{r}_j) = -\vec{F}_{ij} = \vec{\nabla}_j V_{ij} = -\vec{\nabla}_{ij} V_{ij}$$

$$= -\vec{\nabla}_{ji} V_{ji}.$$

Consequentemente, $(H3) \Rightarrow (H2)$

e

$$W_{int_{a,b}} = -\sum_i \sum_{j \neq i} \int_a^b \vec{\nabla}_i V_{ij} \cdot d\vec{r}_i$$

$$= -\int_a^b \Big(\vec{\nabla}_1 V_{12} \cdot d\vec{r}_1 + \vec{\nabla}_1 V_{13} \cdot d\vec{r}_1 + \dots$$

$$+ \vec{\nabla}_2 V_{21} \cdot d\vec{r}_2 + \vec{\nabla}_2 V_{23} \cdot d\vec{r}_2 + \dots$$

$$+ \vec{\nabla}_3 V_{31} \cdot d\vec{r}_3 + \vec{\nabla}_3 V_{32} \cdot d\vec{r}_3 + \dots$$

$$+ \dots \Big)$$

$$= -\int_a^b \left(\vec{\nabla}_{12} V_{12} \cdot d\vec{r}_{12} + \vec{\nabla}_{13} V_{13} \cdot d\vec{r}_{13} + \dots + \vec{\nabla}_{23} V_{23} \cdot d\vec{r}_{23} + \dots \right.$$

$$= -\sum_i \sum_{j>i} \int_a^b \vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij}$$

$$= -\frac{1}{2} \sum_i \sum_{j \neq i} \int_a^b \vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij} = -\frac{1}{2} \sum_i \sum_{j \neq i} V_{ij} \Big|_a^b .$$

HIPÓTESE 4:

$$\underbrace{\sum_i \vec{F}_i^{(v)} \cdot \delta \vec{r}_i = 0}$$

trabalho virtual das forças de vínculo é nulo:

Automaticamente satisfeito para vínculos de corpos rígidos ($d\vec{r}_{ij} \perp \vec{r}_{ij}$) e para vínculos ortogonais às possíveis trajetórias das respectivas partículas.

Mas e: → a tensão numa máquina de Atwood, p.ex.?
↘ $\sum_i$!

→ o atrito no disco rolando, p.ex.?
↘ em geral atritos, tratados macroscopicamente, de fato não satisfazem (H4).
→ (Ter isso em mente as we go!)

• 2ª lei de Newton:

$$\vec{F}_{R_i} - \dot{\vec{p}}_i = \vec{0} \quad \forall\, i = 1, \dots, N$$

$$\therefore \sum_i \left(\vec{F}_{R_i} - \dot{\vec{p}}_i \right) \cdot \delta \vec{r}_i = 0$$

$$\Rightarrow \sum_i \left(\vec{F}_i + \vec{F}_i^{(v)} - \dot{\vec{p}}_i \right) \cdot \delta \vec{r}_i \overset{(H4)}{=} \underbrace{\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i \right) \cdot \delta \vec{r}_i = 0}_{\text{Princípio de D'Alembert}}$$

• Princípio de D'Alembert.

$$\underbrace{\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i\right) \cdot \delta\vec{r}_i = 0}$$

Vínculos ⟹ $\delta\vec{r}_i$ não são independentes

Quem são independentes? COORDENADAS GENERALIZADAS

q_j

• Reescrevendo o Princípio de D'Alembert em termos das coordenadas generalizadas:

- Deslocamentos virtuais: $\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\,\delta q_j$

∴ Trabalho virtual:

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{N}\sum_{j=1}^{3N-S} \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}\,\delta q_j$$

$$= \sum_{j=1}^{3N-S} \underbrace{\left(\sum_i \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_j}\right)}_{=:\,Q_j}\,\delta q_j$$

$=: Q_j$: "FORÇA GENERALIZADA"

– Resta reescrever a soma

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i$$ em termos das coordenadas gener-
lizadas

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \left(\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j\right)$$

$$= \sum_{i,j} m_i \underbrace{\left(\ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)} \delta q_j$$

$$= \frac{d\dot{\vec{r}}_i}{dt} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \underbrace{\frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)}_{(A)} - \dot{\vec{r}}_i \, \underbrace{\frac{d}{dt}}_{(}$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_k \partial q_j}\frac{dq_k}{dt} + \frac{\partial^2 \vec{r}_i}{\partial t\, \partial q_j}$$

$$= \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j\, \partial t}$$

HIPÓTESE 5: q's e $\dot{q}$'s (i.e. coordenadas e velocidades generalizadas) todas independentes entre si!

$$= \frac{\partial}{\partial q_j}\underbrace{\left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)}_{= \frac{d\vec{r}_i}{dt} = \dot{\vec{r}}_i} = \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

notando-se que, com (H5), tem-se

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$, que podemos usar no termo (A

$$\therefore \quad \sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{i,j} m_i \left(\frac{d}{dt} \underbrace{\left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} \right)}_{= \frac{1}{2} \frac{\partial}{\partial \dot{q}_j} (\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i)} - \underbrace{\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j}}_{= \frac{1}{2} \frac{\partial}{\partial q_j} (\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i)} \right) \delta q_j$$

$$\Rightarrow \sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{i,j} m_i \left[\left(\frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} - \frac{1}{2} \frac{\partial}{\partial q_j} \right) \left(\dot{\vec{r}}_i^{\,2} \right) \right] \delta q_j$$

$$\overset{(H1)^*}{=} \sum_j \left(\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j} \right) \left(\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} \right) \delta q_j$$

$$= \sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j .$$

Juntando com o trabalho virtual de volta no princípio de D'Alembert:

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = \sum_j \left(Q_j - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial T}{\partial q_j} \right) \delta q_j = 0$$

Deslocamentos virtuais arbitrários e independentes entre si:

$$\Rightarrow \boxed{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \quad \forall j }$$

$(H1)^*$: $\frac{\partial m_i}{\partial q_j} = 0$ e $\frac{\partial m_i}{\partial \dot{q}_j} = 0$

* Forças conservativas e não-conservativas:

$$\forall i: \quad \vec{F}_i = \vec{F}_i^{(cons)} + \vec{F}_i^{(\tilde{n}cons)}$$

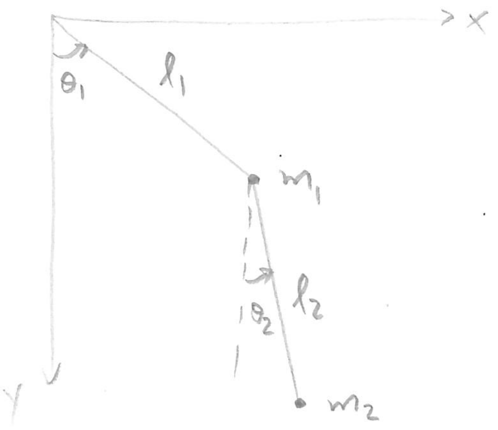
$$= -\vec{\nabla}_i V + \vec{F}_i^{(\tilde{n}cons)}$$

$$\Rightarrow \forall j: \quad Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= \sum_i \left(-\vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \vec{F}_i^{(\tilde{n}cons)} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right)$$

$$= -\frac{\partial V}{\partial q_j} + \underbrace{\sum_i \vec{F}_i^{(\tilde{n}cons)} \cdot \frac{\partial \vec{r}_i}{\partial q_j}}_{=: Q_j^{(\tilde{n}cons)}}$$

EXEMPLO 1: Pêndulo duplo



$$x_1^2 + y_1^2 = l_1^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$$

$$(x_1, y_1, x_2, y_2) \longmapsto (\theta_1, \theta_2)$$

EXEMPLO 2: $\underbrace{\text{Disco}}_{\text{raio } r}$ $\underbrace{\text{rolando sem deslizar}}$ num plano horizon[tal]

$$r\dot{\phi} = |\vec{V}| = |\dot{\vec{R}}| \qquad (\mathrm{I})$$

$$\vec{R} = x\,\hat{i} + y\,\hat{j}$$

$$\dot{\vec{R}} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} \qquad (\mathrm{II})$$

$$= |\dot{\vec{R}}|\cos\theta\,\hat{i} + |\dot{\vec{R}}|\,\mathrm{sen}\,\theta\,\hat{j} \qquad (\mathrm{III})$$

De (I), (II) e (III):

$$\begin{cases} \dot{x} - r\dot{\phi}\cos\theta = 0 \\ \dot{y} - r\dot{\phi}\,\mathrm{sen}\,\theta = 0 \end{cases}$$

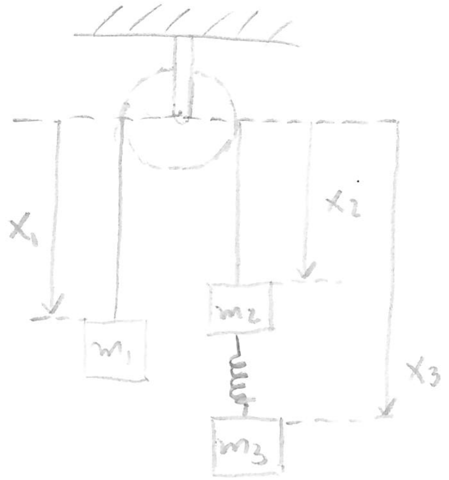
ou, equivalentemente:

$$\begin{cases} dx = r\cos\theta\, d\phi \\ dy = r\,\mathrm{sen}\,\theta\, d\phi \end{cases}$$

$$(x, y, \theta, \phi) \longmapsto ?$$

Exemplo de vínculos diferenciais não integráveis: "restringem os deslocamentos possíveis do sistema mas não impõem quaisquer limites às configurações possíveis, de modo que a priori todas as configurações são acessíveis." (Nivaldo Lemos, p. 13).

EXEMPLO 3: Sistema de polia, blocos e mola



$$x_1 + x_2 = \ell$$

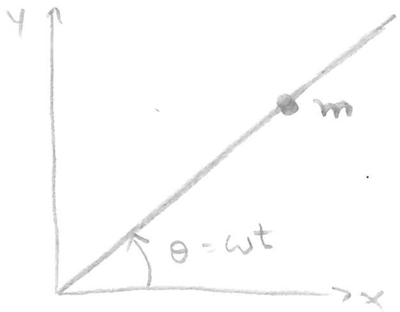
$$x_3 = x_2 + d_0 + d$$

$$(x_1, x_2, x_3, d) \longmapsto (x_1, x_3)$$

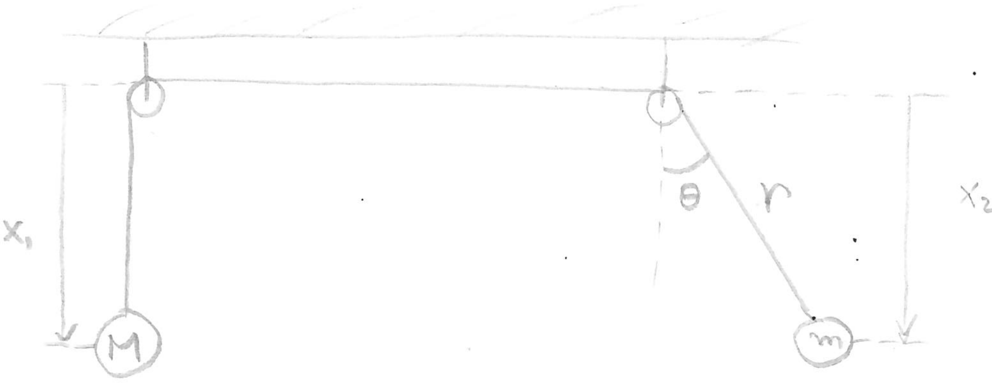
$$\longmapsto (x_2, x_3)$$

$$\longmapsto (x_2, d)$$

EXEMPLO 4: Conta deslizando ao longo de uma haste girante



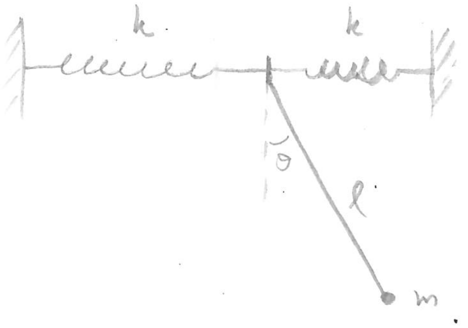
EXEMPLO 5: Máquina de Atwood oscilante



$$\left.\begin{aligned} x_1 + r &= \ell \\ x_2 &= r\cos\theta \\ y_2 &= r\,\mathrm{sen}\,\theta \end{aligned}\right\} \quad x_1 + x_2 \sec\theta = \ell$$

$$(x_1, x_2, y_2) \longmapsto (r, \theta)$$

EXEMPLO 6:



$$l_1 + d_1 + l_2 + d_2 = l_0$$

$$l_1 + d_1 = x_0$$

$$x^2 + y^2 = l^2$$

$(x, y, d_1, d_2, x_0) \longmapsto (x_0, \theta)$, p.ex. (Nivaldo Lemos, ex. 1.